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Demostrar que:

$$\int_{-\infty}^{\infty} dx \frac{m^2}{2} \phi^2 = \frac{1}{8} \int_{-\infty}^{\infty} \frac{dk}{2\pi\omega^2} m^2 \left[a_k a_k^* + a_k^* a_k + a_k a_{-k} e^{-i2\omega t} + a_k^* a_{-k}^* e^{i2\omega t} \right]$$

siendo

$$\phi_{(x,t)} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi\omega} \left[a_k e^{-i(\omega t - kx)} + a_k^* e^{i(\omega t - kx)} \right] = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi\omega} f_{(k,x,t)}$$

$$\phi^2 = \frac{1}{4} \int_{-\infty}^{\infty} \frac{dk}{2\pi\omega} f(k,x,t) \underbrace{\int_{-\infty}^{\infty} \frac{dk}{2\pi\omega} f(k,x,t)}_{k \to q \; ; \; \omega \to \Omega = \sqrt{q^2 + m^2}}$$

$$\phi^2 = \frac{1}{4} \int_{-\infty}^{\infty} \frac{dk}{2\pi\omega} f_{(k,x,t)} \int_{-\infty}^{\infty} \frac{dq}{2\pi\Omega} f(q,x,t)$$

$$= \frac{1}{4} \int_{-\infty}^{\infty} \frac{dk}{2\pi\omega} \int_{-\infty}^{\infty} \frac{dq}{2\pi\Omega} f(k,x,t) f(q,x,t)$$

$$\phi^2 = \frac{1}{4} \int_{-\infty}^{\infty} \frac{dk}{2\pi\omega} \int_{-\infty}^{\infty} \frac{dq}{2\pi\Omega} \left[a_k e^{-i(\omega t - kx)} + a_k^* e^{i(\omega t - kx)} \right] \left[a_q e^{-i(\Omega t - qx)} + a_q^* e^{i(\Omega t - qx)} \right]$$

$$\phi^2 = \frac{1}{4} \int_{-\infty}^{\infty} \frac{dk}{2\pi\omega} \int_{-\infty}^{\infty} \frac{dq}{2\pi\Omega} \Big[a_k a_q e^{-i(\omega t - kx)} e^{-i(\Omega t - qx)} +$$

$$+ a_k a_q^* e^{-i(\omega t - kx)} e^{i(\Omega t - qx)} +$$

$$+ a_k^* a_q e^{i(\omega t - kx)} e^{-i(\Omega t - qx)} +$$

$$+ a_k^* a_q^* e^{i(\omega t - kx)} e^{i(\Omega t - qx)} \Big]$$

$$\phi^2 = \frac{1}{4} \int_{-\infty}^{\infty} \frac{dk}{2\pi\omega} \int_{-\infty}^{\infty} \frac{dq}{2\pi\Omega} g(k,q,x,t)$$

$$\int_{-\infty}^{\infty} dx\, \phi^2 = \frac{1}{4} \int_{-\infty}^{\infty} \frac{dk}{2\pi\omega} \int_{-\infty}^{\infty} \frac{dq}{2\pi\Omega} \int_{-\infty}^{\infty} dx\, g(k,q,x,t)$$

NOTA

$$\int_{-\infty}^{\infty} dx\, g_{(k,q,x,t)} = \int_{-\infty}^{\infty} dx \Big[a_k a_q\, e^{-i((\omega+\Omega)t-(k+q)x)} +$$
$$+ a_k a_q^* \, e^{-i((\omega-\Omega)t-(k-q)x)} +$$
$$+ a_k^* a_q \, e^{i((\omega-\Omega)t-(k-q)x)} +$$
$$+ a_k^* a_q^* \, e^{i((\omega+\Omega)t-(k+q)x)} \Big]$$

$$\int_{-\infty}^{\infty} dx\, a_k a_q\, e^{-i(\omega+\Omega)t}\, e^{i(k+q)x} = a_k a_q\, e^{-i(\omega+\Omega)t} \underbrace{\int_{-\infty}^{\infty} dx\, e^{i(k+q)x}}_{2\pi\,\delta(k+q)}$$

pero $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, dx$

y $\delta(x)$ es PAR

$$= a_k a_q\, e^{-i(\omega+\Omega)t}\; 2\pi\,\delta(k+q)$$

$$\int_{-\infty}^{\infty} dx\, a_k a_q^*\, e^{-i(\omega-\Omega)t}\, e^{i(k-q)x} = a_k a_q^*\, e^{-i(\omega-\Omega)t}\; 2\pi\,\delta(k-q)$$

$$\int_{-\infty}^{\infty} dx\, a_k^* a_q\, e^{i(\omega-\Omega)t}\, e^{-i(k-q)x} = a_k^* a_q\, e^{i(\omega-\Omega)t}\; 2\pi\,\delta(k-q)$$

$$\int_{-\infty}^{\infty} dx\, a_k^* a_q^*\, e^{i(\omega+\Omega)t}\, e^{-i(k+q)x} = a_k^* a_q^*\, e^{i(\omega+\Omega)t}\; 2\pi\,\delta(k+q)$$

Entonces:

$$\int_{-\infty}^{\infty} dx\, \phi^2 = \frac{1}{4}\int_{-\infty}^{\infty}\frac{dk}{2\pi\omega}\int_{-\infty}^{\infty}\frac{dq}{2\pi\Omega}\Big[a_k a_q\, e^{-i(\omega+\Omega)t}\, 2\pi\,\delta(k+q) +$$
$$+ a_k a_q^*\, e^{-i(\omega-\Omega)t}\, 2\pi\,\delta(k-q) +$$
$$+ a_k^* a_q\, e^{i(\omega-\Omega)t}\, 2\pi\,\delta(k-q) +$$
$$+ a_k^* a_q^*\, e^{i(\omega+\Omega)t}\, 2\pi\,\delta(k+q) \Big]$$

$$\int_{-\infty}^{\infty} \frac{dq}{\underset{\uparrow \atop \omega}{2\pi\Omega}}\, a_k \underset{\uparrow \atop -k}{a_q}\, e^{-i(\underset{\uparrow \atop \omega}{\omega+\Omega})t}\; \underbrace{2\pi\,\delta(k+q)}_{\to q=-k,\ \Omega=\omega} = a_k a_{-k}\, e^{-i2\omega t}\; 2\pi\, \frac{1}{2\pi\omega}$$

$$= \frac{1}{\omega}\, a_k a_{-k}\, e^{-i2\omega t}$$

$$\int_{-\infty}^{\infty} \frac{dq}{\underset{\uparrow\,\omega}{2\pi\Omega}}\, a_k \underset{\uparrow\,k}{a_q^*}\, e^{\underset{\uparrow\,\omega}{-i(\omega-\Omega)t}}\, 2\pi\, \delta\underbrace{(k-q)}_{k=q} = \frac{1}{2\pi\omega}\, a_k a_k^*\, 2\pi = \frac{1}{\omega} a_k a_k^*$$

$$\int_{-\infty}^{\infty} \frac{dq}{2\pi\Omega}\, a_k^* a_q\, e^{i(\omega-\Omega)t}\, 2\pi\, \delta(k-q) = \frac{1}{2\pi\omega}\, a_k^* a_k\, 2\pi = \frac{1}{\omega} a_k^* a_k$$

$$\int_{-\infty}^{\infty} \frac{dq}{2\pi\Omega}\, a_k^* a_q^*\, e^{i(\omega+\Omega)t}\, 2\pi\, \delta(k+q) = \frac{1}{2\pi\omega}\, a_k^* a_{-k}^*\, e^{i2\omega t}\, 2\pi$$

$$= \frac{1}{\omega} a_k^* a_{-k}^* e^{i2\omega t}$$

$$\int_{-\infty}^{\infty} dx\, \phi^2 = \frac{1}{4}\int_{-\infty}^{\infty} \frac{dk}{2\pi\omega}\left[\frac{1}{\omega} a_k a_{-k} e^{-i2\omega t} + \frac{1}{\omega} a_k a_k^* + \frac{1}{\omega} a_k^* a_k + \frac{1}{\omega} a_k^* a_{-k}^* e^{i2\omega t}\right]$$

multiplicando por $m^2/2$

$$\int_{-\infty}^{\infty} dx\, \frac{m^2}{2}\phi^2 = \frac{1}{8}\int_{-\infty}^{\infty} \frac{dk}{2\pi\omega^2}\, m^2 \left[a_k a_k^* + a_k^* a_k + a_k a_{-k} e^{-i2\omega t} + a_k^* a_k^* e^{i2\omega t}\right]$$

Q.E.D.